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COMMENT

On a new class of planar fractals: the Pascal–Sierpinski gaskets

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Abstract. A new class of planar fractals called the Pascal-Sierpinski gaskets is described, of which the well known Sierpinski gasket is a special case. Some of these gaskets are true Mandelbrot fractals possessing non-integral dimensions as well as self-similarity; the remaining ones are not self-similar, but appear to have non-integral dimensions.

Physics, in general, until recently has been greatly limited in its scope by the use of Euclidean geometry whereby all bodies possess integral dimensionality ranging from 0 to 3. But, to quote from the cover flap of Mandelbrot's celebrated work (1983), 'all clouds are not spheres, mountains are not cones, and lightning does not travel in a straight line'. The forms exhibited by nature differ so much from Euclidean objects that an entirely new geometry has had to be applied, principally by Mandelbrot himself (1983) but also by several others (e.g. Lovejoy 1982, Berry 1982, Walker and Jakeman 1982, Yehoda and Messier 1985). The relatively huge amount of effort put forth in the last few years has resulted in the identification of several natural shapes which are now known as *fractals.*

Fractals are bodies characterised by several properties. Among the more notable ones are their possession of non-integral or fractional dimensionality and self-similarity. Natural fractals are found to be scale-invariant over several length scales (Yehoda and Messier 1985), whereas strictly geometric fractals are scale-invariant over all possible appropriate scales. An adequate discussion of these and other properties of fractals can be found elsewhere (Mandelbrot 1983), for which reason they will not be reviewed in any detail here.

Though natural fractals occur in any landscape and in such 'random' structures as thin films and are, as a result, probably more familiar, strictly geometric fractals are not of solely academic interest. The Sierpinski gasket (fractal dimension $= \log 3/\log 2$), which has been paid a great deal of attention by Mandelbrot (1983), is a geometric fractal of interest in percolation morphology studies. In studying percolation through lattices, it was observed by Gefen *et al* (1980, 1981) that the branching structure of the Sierpinski gasket (and of its three- and four-dimensional analogues) proves to be a promising model of the structure of the cluster backbones. Recent work on the Sierpinski gasket has also been reported by Stephen (1981), Rammal and Toulouse (1982) as well as by Alexander and Orbach (1982).

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In this comment the authors are primarily concerned with describing a new class of geometric fractals which can be derived from the well known Pascal's triangle, and of which the Sierpinski gasket is a special case. It will be shown here that with the complete (i.e. infinitely large) Pascal's triangle as a base, fractal surfaces ranging up to a dimension of 2.0 can be constructed; the unaltered Pascal's triangle being of exactly 2.0 dimensionality. Several other structures also emerge which do not possess self-similarity in a strict sense, but which appear to have non-integral dimensionality.

Pascal's triangle constitutes an equiangular triangular grid whose rows shall be labelled by $n=1,2,3,\ldots,$ each row containing n nodes $\{n, p_n\}$, $p_n=1,2,3,\ldots,n$. Attached to each of the nodes on this triangular grid is a number ${}^nC_{p_n} = (n-1)!/[(n-p_n)!(p_n-1)!]$, which are nothing but the p_nth coefficients of the binomial expansion of $(x+y)^{n-1}$. As is readily seen, the triangular grid thus formed is infinitely large, the numbers ${}^nC_{p_n}$ growing without bounds as n increases. Incidentally, ${}^nC_1 = {}^nC_n = 1$. An extensive table of these numbers can be found in a handbook by Abramowitz and Stegun (1970).

The fractals to be described here can be evolved from any truncated Pascal's triangle of a suitably large size and they shall be referred to as the Pascal–Sierpinski gaskets hereafter. The algorithm for generating them now follows.

First of all, a label ${}^nL_{p_n}$ is attached to each of the nodes $\{n, p_n\}$. Any integer $N>1$ may now be chosen, and the labels ${}^nL_{p_n}$ are defined by

$$ {}^nL_{p_n} = 1 \qquad \text{if } {}^nC_{p_n} \text{ is } \textit{not exactly divisible} \text{ by } N \tag{1a} $$

and

$$ {}^nL_{p_n} = 0 \qquad \text{if } {}^nC_{p_n} \text{ is } \textit{exactly divisible} \text{ by } N. \tag{1b} $$

The resulting map of the labels ${}^nL_{p_n}$ superimposed on the triangular grid constitutes a Pascal–Sierpinski gasket of order N. It will now be shown that the gaskets thus formed may be considered as fractals.

Let each of the nodes in this map be treated as having a mass ${}^nL_{p_n}$. The grid will contain several voids and several filled areas, and figures 1–5 show the generated gaskets when $1 \leqslant n \leqslant 64$, while N varies from 2 to 6.

Inspection of these gaskets shows that when N is a prime number, they are self-similar in the Mandelbrot sense. Their fractal (similarity) dimension d_N can then be easily computed to be

$$ d_N = \log\{N(N+1)/2\}/\log\{N\} \qquad N \text{ prime} \tag{2} $$

which turns out to be 1.584 9625 for $N=2$, and

$$ \lim_{N\to\infty}\{d_N\} = 2 \qquad N \text{ prime.} \tag{3} $$

Furthermore, for $N=2$, the gasket generated is nothing but the Sierpinski gasket (Mandelbrot 1983).

Next come the cases when N is an integral power of a prime number. In such cases, as in figure 3 for $N=4=2^2$, visual inspection alone suffices to show that the resulting gaskets are self-similar. However, a simple formula like (2) for the fractal dimension d_N could not be deduced by the authors for these cases.

Finally we look at the cases for all other values of $N>1$, when N is neither a prime nor an integral power of a prime, as in figure 6 for $N=6=2\times 3$. Visual inspection extended up to $n=198$ rows did not show any self-similarity in these gaskets. However,

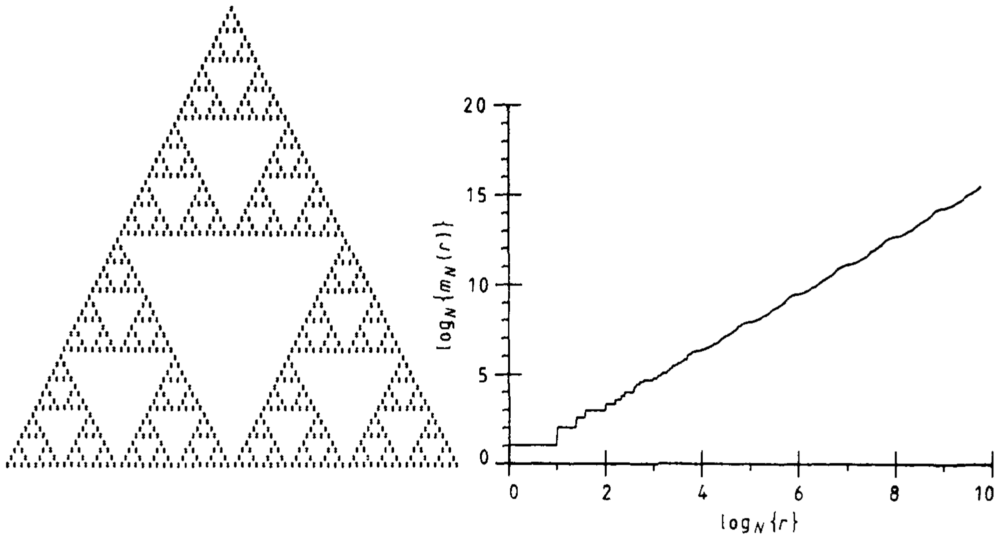

Figure 1. The Pascal-Sierpinski gasket of order $N = 2$. Shown are the first $n = 64$ rows of the gasket. The graph alongside plots the mass-radius variation for this gasket when the first $n = 1000$ rows have been considered. The coordinate ranges are $0 \leqslant \log_N\{m_N(r)\} \leqslant 20.0$ and $0 \leqslant \log_N\{r\} \leqslant 10.0$.

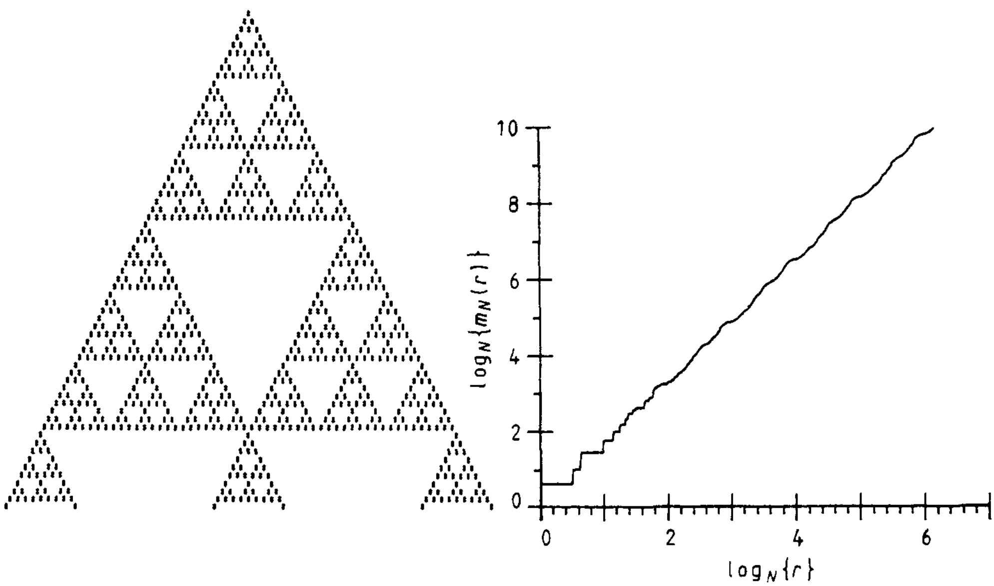

Figure 2. The Pascal-Sierpinski gasket of order $N = 3$. Shown are the first $n = 64$ rows of the gasket. The graph alongside plots the mass-radius variation for this gasket when the first $n = 1000$ rows have been considered. The coordinate ranges are $0 \leqslant \log_N\{m_N(r)\} \leqslant 10.0$ and $0 \leqslant \log_N\{r\} \leqslant 7.0$.

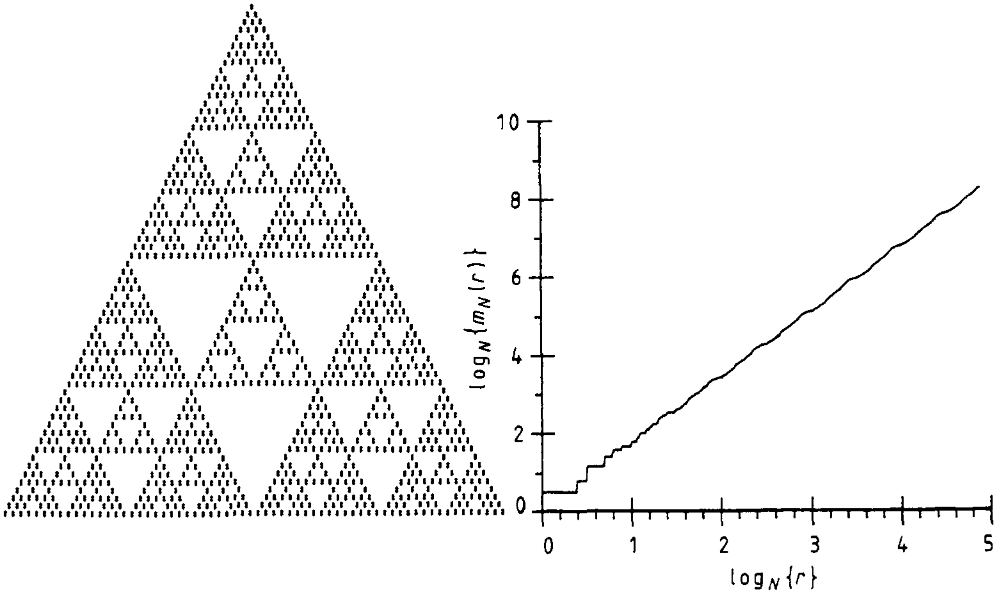

Figure 3. The Pascal-Sierpinski gasket of order $N = 4$. Shown are the first $n = 64$ rows of the gasket. The graph alongside plots the mass-radius variation for this gasket when the first $n = 1000$ rows have been considered. The coordinate ranges are $0 \leq \log_N\{m_N(r)\} \leq 10.0$ and $0 \leq \log_N\{r\} \leq 5.0$.

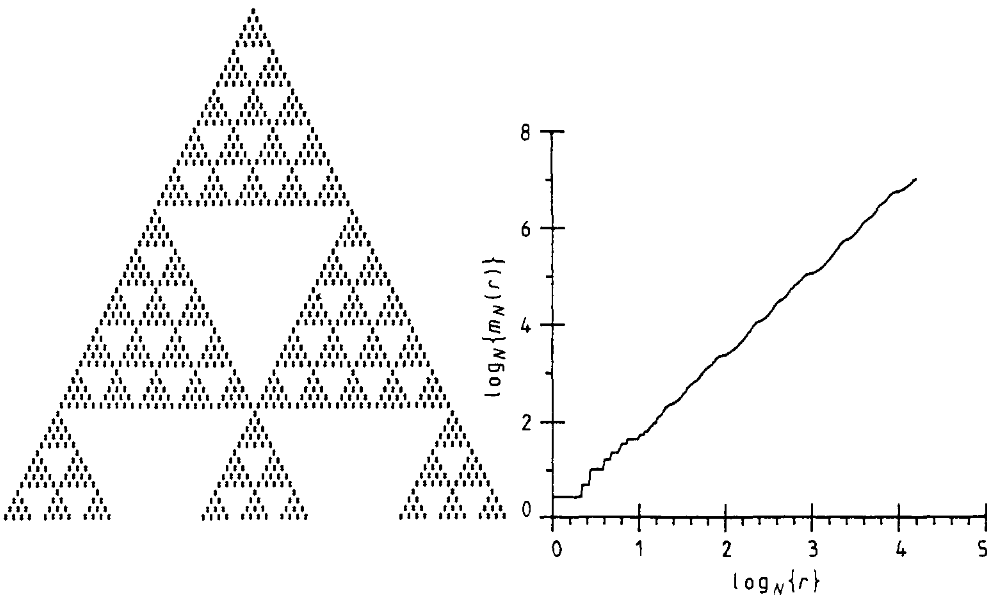

Figure 4. The Pascal-Sierpinski gasket of order $N = 5$. Shown are the first $n = 64$ rows of the gasket. The graph alongside plots the mass-radius variation for this gasket when the first $n = 1000$ rows have been considered. The coordinate ranges are $0 \leq \log_N\{m_N(r)\} \leq 8.0$ and $0 \leq \log_N\{r\} \leq 5.0$.

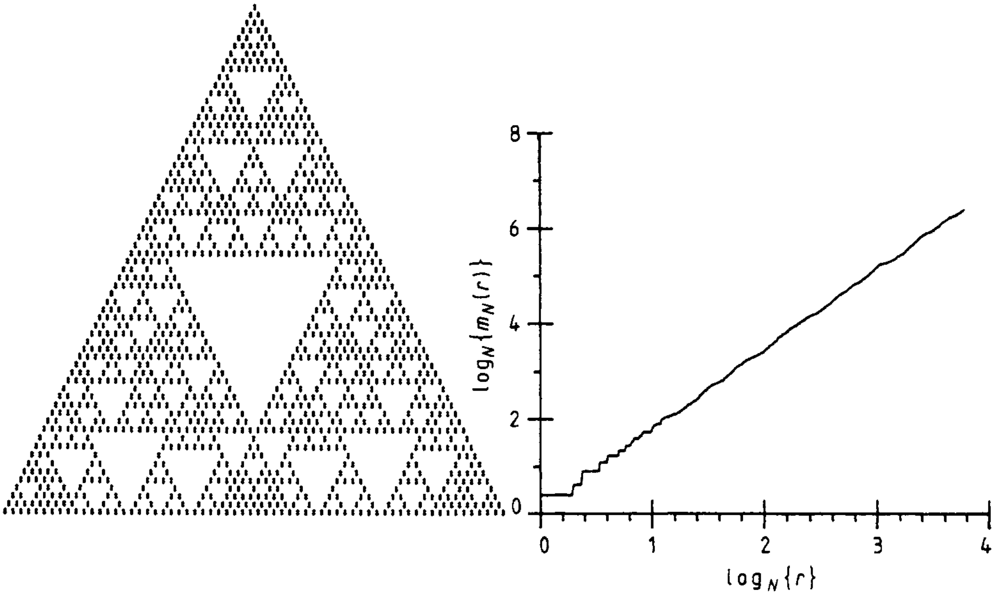

Figure 5. The Pascal-Sierpinski gasket of order $N=6$. Shown are the first $n=64$ rows of the gasket. The graph alongside plots the mass-radius variation for this gasket when the first $n=1000$ rows have been considered. The coordinate ranges are $0 \leq \log_N\{m_N(r)\} \leq 8.0$ and $0 \leq \log_N\{r\} \leq 4.0$.

if the factorisation

$$N = \Pi_i(j_i)^{k_i} = \Pi_i N_i \tag{4}$$

is made, where all k_i are real positive integers and all j_i are primes, then the gasket of order N can be synthesised from the gaskets of orders N_i by the operation

$$\text{gasket}\{N\} = \text{gasket}\{N_1\} \oplus \text{gasket}\{N_2\} \oplus \text{gasket}\{N_3\} \oplus \ldots \tag{5}$$

where $\oplus$ denotes an OR operation (Baron and Piccirilli 1967). The operation OR between any two gaskets of orders N_i and N_j is performed by overlaying one on top of the other such that their like-numbered nodes coincide exactly. Then for each node $\{n, p_n\}$ of the resulting overlay, the resulting label is given by

$$^nL_{p_n}\{N_i \oplus N_j\} = {}^nL_{p_n}\{N_i\} \oplus {}^nL_{p_n}\{N_j\} \tag{6a}$$

where the familiar rules of Boolean algebra

$$1 \oplus 1 = 1 \qquad 1 \oplus 0 = 1, \qquad 0 \oplus 1 = 1 \qquad 0 \oplus 0 = 0 \tag{6b}$$

apply. Because of (5) it is conjectured that this last class of the Pascal-Sierpinski gaskets should also possess fractal dimensions, though they are not self-similar.

Since a fairly general method of determining the fractal dimension d_N was needed, in view of the fact that the formula (2) is applicable only for prime N, the mass-radius fractal dimension D_N was determined for the gaskets of figures 1-5. As has been mentioned earlier, each of the nodes in the gaskets was given a mass $^nL_{p_n}$, while the node $\{1, 1\}$ was treated as the datum point for measurement of the distance r. Since the triangular grid is equiangular and its rows equispaced, the nearest-neighbour distance between the nodes was taken to be unity. It may be noted that if these gaskets are not fractal, then $D_N = 2.0$ should result from such a procedure; if, however, they are, then D_N should converge to d_N, $1.0 < d_N \leqslant 2.0$, as their truncation levels are increased. Measurements were made by sweeping out 60° wide symmetric arcs of increasing radii r from the datum node, the sum of the labels $^nL_{p_n}$ covered in the sector giving the mass $m_N(r)$ of the sectoral plates thus formed. These computations were made for all gaskets or orders $N = 2$–5, with the number of rows set to be $n \leqslant 198$, 500 and 1000. Also shown in figures 1–5 are the plots of $\log_N\{m_N(r)\}$ against $\log_N\{r\}$ for the gaskets truncated at $n = 1000$. Finally, through each of these plots a straight line was fitted, using a least-squares curve-fitting procedure (Worthing and Geffner 1948), for reasonably high values of $\log_N\{r\}$. The slope of this line is the mass-radius dimension D_N.

It is observed from table 1, where the computed D_N are given, that as the truncation level of the Pascal-Sierpinski gaskets is increased, the dimension D_N appears to be converging towards the 'expected' fractal dimension d_N for prime N. Likewise, even for the other cases of non-prime N, these data clearly suggest the existence of a fractal dimension.

The number of massless points for each of the five gaskets considered at levels of truncation $n = 198$, 500 and 1000 are shown in table 2. A comparison of tables 1 and 2 reveals that $D_4 < D_5 < D_6$, even though the order 5 gasket contains more massless nodes than the gaskets of orders 4 or 6. Normally, one would expect that a larger number of massless nodes should give rise to a smaller D_N; yet this is not the case here. And, although the gasket of order 6 is not self-similar, the other two are. This feature, that the larger number of voids is not related to a smaller mass-radius dimension, appears to be due to the differences in the textures of the pertinent gaskets (Mandelbrot 1983).

Table 1. Mass-radius dimension D_N.

	D_N			
Order N	$n \leqslant 198$	$n \leqslant 500$	$n \leqslant 1000$	d_N
2	1.5681	1.5716	1.5738	1.584 96
3	1.6134	1.6138	1.6218	1.630 93
4	1.6567	1.6617	1.6688	
5	1.6645	1.6688	1.6691	1.682 61
6	1.6780	1.6712	1.6693	

Table 2. Number of massless nodes against total number of nodes.

	Number of massless nodes/total number of nodes		
Order N	$n \leqslant 198$	$n \leqslant 500$	$n \leqslant 1000$
2	15 996/19 701	107 871/125 250	448 363/500 500
3	14 949/19 701	101 742/125 250	437 420/500 500
4	12 953/19 701	91 495/125 250	393 442/500 500
5	13 896/19 701	91 500/125 250	409 375/500 500
6	12 393/19 701	88 766/125 250	395 000/500 500

In summary, a class of gaskets named after Pascal and Sierpinski has been described here, of which the Sierpinski gasket (Mandelbrot 1983) is a special case. It is shown that the mass-radius dimensions of these gaskets are fractions greater than unity but less than 2.0. Furthermore, some of these gaskets are self-similar and thus form true Mandelbrot fractals. Conceivably, these gaskets are of use in modelling percolation clusters.

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